

# Frequency localized regularity criteria for the 3D Navier-Stokes equations

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## Abstract

Two regularity criteria are established to highlight which Littlewood-Paley frequencies play an essential role in possible singularity formation in a Leray-Hopf weak solution to the Navier-Stokes equations in three spatial dimensions. One of these is a frequency localized refinement of known Ladyzhenskaya-Prodi-Serrin-type regularity criteria restricted to a finite window of frequencies the lower bound of which diverges to  $+\infty$  as  $t$  approaches an initial singular time.

## 1 Introduction

The Navier-Stokes equations governing the evolution of a viscous, incompressible flow's velocity field  $u$  in  $\mathbb{R}^3 \times (0, T)$  read

$$\begin{aligned} \partial_t u + u \cdot \nabla u &= -\nabla p + \nu \Delta u + f & \text{in } \mathbb{R}^3 \times (0, T) \\ \nabla \cdot u &= 0 & \text{in } \mathbb{R}^3 \times (0, T), \end{aligned} \quad (3D \text{ NSE})$$

where  $\nu$  is the viscosity coefficient,  $p$  is the pressure, and  $f$  is the forcing. For convenience we take  $f$  to be zero and set  $\nu = 1$ . The flow evolves from an initial vector field  $u_0$  taken in an appropriate function space.

The regularity of Leray-Hopf weak solutions (i.e. distributional solutions for  $u_0 \in L^2$  that satisfy the global energy inequality and belong to  $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$  for any  $T > 0$ ) remains an open problem. The best results available rely on critical quantities being finite, that is quantities which are invariant given the natural scaling associated with the Navier-Stokes equations. In this note we provide several regularity criteria which highlight the essential role of high frequencies in a possibly singular Leray-Hopf weak solution.

Frequencies are interpreted in the Littlewood-Paley sense. Let  $\lambda_j = 2^j$  for  $j \in \mathbb{Z}$  be measured in inverse length scales and let  $B_r$  denote the ball of radius  $r$  centered at the origin. Fix a non-negative, radial cut-off function  $\chi \in C_0^\infty(B_1)$  so that  $\chi(\xi) = 1$  for all  $\xi \in B_{1/2}$ . Let  $\phi(\xi) = \chi(\lambda_1^{-1}\xi) - \chi(\xi)$  and  $\phi_j(\xi) = \phi(\lambda_j^{-1})(\xi)$ . Suppose that  $u$  is a vector field

of tempered distributions and let  $\Delta_j u = \mathcal{F}^{-1} \phi_j * u$  for  $j \in \mathbb{N}$  and  $\Delta_{-1} = \mathcal{F}^{-1} \chi * u$ . Then,  $u$  can be written as

$$u = \sum_{j \geq -1} \Delta_j u.$$

If  $\mathcal{F}^{-1} \phi_j * u \rightarrow 0$  as  $j \rightarrow -\infty$  in the space of tempered distributions, then for  $j \in \mathbb{Z}$  we define  $\Delta_j u = \mathcal{F}^{-1} \phi_j * u$  and have

$$u = \sum_{j \in \mathbb{Z}} \Delta_j u.$$

For  $s \in \mathbb{R}$ ,  $1 \leq p, q \leq \infty$  the homogeneous Besov spaces include tempered distributions modulo polynomials for which the norm

$$\|u\|_{\dot{B}_{p,q}^s} := \begin{cases} \left( \sum_{j \in \mathbb{Z}} (\lambda_j^s \|\Delta_j u\|_{L^p(\mathbb{R}^n)})^q \right)^{1/q} & \text{if } q < \infty, \\ \sup_{j \in \mathbb{Z}} \lambda_j^s \|\Delta_j u\|_{L^p(\mathbb{R}^n)} & \text{if } q = \infty \end{cases},$$

is finite. See [2] for more details.

Given a Leray-Hopf weak solution  $u$  that belongs to  $C(0, T; \dot{B}_{\infty, \infty}^{-\epsilon})$  for some  $\epsilon$  in  $(0, 1)$ , we define the following upper and lower endpoint frequencies: for  $t$  in  $(0, T)$  let

$$J_{high}(t) = \log_2 \left[ c_1 \|u(t)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}}^{1/(1-\epsilon)} \right], \quad (1)$$

and

$$J_{low}(t) = \log_2 \left[ \left( c_2 \frac{\|u(t)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}}}{\|u\|_{L^\infty(0, T; L^2)}} \right)^{2/(3-2\epsilon)} \right], \quad (2)$$

where  $c_1$  and  $c_2$  are universal constants (their values will become clear in Section 2). Our first regularity criterion shows  $J_{low}$  and  $J_{high}$  determine the Littlewood-Paley frequencies which, if well behaved at a finite number of times prior to a possible blow-up time, prevent singularity formation.

**Theorem 1.** Fix  $\epsilon \in (0, 1)$  and  $T > 0$ , and assume that  $u \in C(0, T; \dot{B}_{\infty, \infty}^{-\epsilon})$  is a Leray-Hopf weak solution to 3D NSE on  $[0, T]$ . If there exists  $t_0 \in (0, T)$  such that

$$\sup_{J_{low}(t_0) \leq j \leq J_{high}(t_0)} \lambda_j^{-\epsilon} \|\Delta_j u(t_i)\|_{L^\infty} \leq \|u(t_0)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}}, \quad (3)$$

where  $\{t_i\}_{i=1}^k \subset (t_0, T)$  is a finite collection of  $k$  times satisfying

$$t_{i+1} - t_i > \left( \frac{c_3}{\|u(t_0)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}}} \right)^{2/(1-\epsilon)} \quad (i = 0, \dots, k-1),$$

and

$$T - t_k < \left( \frac{2c_3}{\|u(t_0)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}}} \right)^{2/(1-\epsilon)}$$

for a universal constant  $c_3$ , then  $u$  can be smoothly extended beyond time  $T$ .

The novelty here is that the solution remains finite provided only a finite range of frequencies remain subdued at a finite number of uniformly spaced times. If  $u$  is not in the energy class then a partial result can be formulated since  $J_{high}$  does not depend on  $\|u\|_{L^\infty(0,T;L^2)}$ . In particular, we just need to replace (3) with

$$\sup_{j \leq J_{high}(t)} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t_i)\|_{L^\infty} \leq \|u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}},$$

and assume  $u$  is the mild solution for  $u_0 \in \dot{B}_{\infty,\infty}^{-\epsilon}$  which is a strong solution on  $[0, T)$  (note that a local-in-time existence theory for mild solution is available in  $\dot{B}_{\infty,\infty}^{-\epsilon}$ ).

Our second result is a refinement of a well known class of regularity criteria (see, e.g., [7]): if  $u$  is a Leray-Hopf weak solution to 3D NSE on  $\mathbb{R}^3 \times [0, T]$  satisfying

$$\int_0^T \|u\|_{L^p}^q dt < \infty,$$

for pairs  $(p, q)$  where  $3 \leq p \leq \infty, 2 \leq q \leq \infty$ , and

$$\frac{2}{q} + \frac{3}{p} = 1,$$

then  $u$  is smooth. This is the Ladyzhenskaya-Prodi-Serrin class for non-endpoint values of  $(p, q)$ . The case  $p = \infty$  is the Beale-Kato-Majda regularity criteria. The case  $p = 3$  was only (relatively) recently proven in [5]. Similar criteria can be formulated for a variety of spaces larger than  $L^p$  when  $p > 3$ . For example, Cheskidov and Shvydkoy give the following Ladyzhenskaya-Prodi-Serrin-type regularity criteria in Besov spaces (see [3]): if  $u$  is a Leray-Hopf solution and  $u \in L^{2/(1-\epsilon)}(0, T; \dot{B}_{\infty,\infty}^{-\epsilon})$ , then  $u$  is regular on  $(0, T]$ . A regularity criterion for weakly time integrable Besov norms in critical classes appears in [1]. In the endpoint case when  $\epsilon = -1$ , smallness is needed either over all frequencies (see [3]) or over high frequencies provided a Beale-Kato-Majda-type bound holds for the projection onto low frequencies (see [4]). Our result is essentially a refinement of the non-endpoint regularity criteria given in [3].

**Theorem 2.** Fix  $\epsilon \in (0, 1)$  and  $T > 0$ , and assume that  $u \in C(0, T; \dot{B}_{\infty,\infty}^{-\epsilon})$  is a Leray-Hopf weak solution to 3D NSE on  $[0, T]$ . If

$$\int_0^T \left( \sup_{J_{low}(t) \leq j \leq J_{high}(t)} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_{L^\infty} \right)^{2/(1-\epsilon)} dt < \infty,$$

then  $u$  is regular on  $(0, T]$ .

Clearly  $J_{high}$  blows up more rapidly than  $J_{low}$  as  $t \rightarrow T^-$  and therefore an increasing number of frequencies are relevant as we approach the possible blow-up time. It is unlikely that this can be improved for weak solutions in supercritical classes like Leray-Hopf solutions. On one hand, the upper cutoff is available because of local well-posedness for the subcritical quantity  $\|u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}$  which suppresses high frequencies at times close to and

after  $t$ . On the other hand, the supercritical quantity  $\|u\|_{L^\infty(0,T;L^2)}$  plays a crucial role in suppressing low frequencies. Any supercritical quantity is sufficient; for example, if we replace  $L^\infty L^2$  with  $L^\infty L^p$  for some  $2 < p < 3$ , then the lower cutoff function is

$$J_{low}(t) = \log_2 \left[ \left( \frac{\|u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}}{c\|u\|_{L^\infty(0,T;L^p)}} \right)^{p/(3-p\epsilon)} \right].$$

Note that  $p/(3-p\epsilon) = 1/(1-\epsilon)$  only when  $p = 3$ , i.e. the exponents in the cutoffs will match only when we reach a critical class  $L^\infty(0,T;L^3)$ .

## 2 Technical lemmas

Local existence of strong solutions for data in the subcritical space  $\dot{B}_{\infty,\infty}^{-\epsilon}$  is known, see [7]. Results in spaces close to  $\dot{B}_{\infty,\infty}^{-\epsilon}$  are given in [6, 9]. Indeed, the proof of [6, Theorem 1] can be modified to show that if  $a \in \dot{B}_{\infty,\infty}^{-\epsilon}$ , then the Navier-Stokes equations have a unique strong solution  $u$  which persists at least until time

$$T_* = \left( \frac{c_0}{\|a\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}} \right)^{2/(1-\epsilon)}, \quad (4)$$

for a universal constant  $c_0$ . Moreover we have

$$\|u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}} \leq c_0 \|a\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}, \quad (5)$$

and

$$t^{1/2} \|\nabla u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}} \leq c_0 \|a\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}, \quad (6)$$

for any  $t \in (0, T_*)$  (the value of  $c_0$  changes from line to line but always represents a universal constant). Since the proof of this is nearly identical to the proof of [6, Theorem 1] it is omitted. Note that by [7, Proposition 3.2], the left hand side of (6) can be replaced by  $t^{1/2} \|u\|_{\dot{B}_{\infty,\infty}^{1-\epsilon}}$ .

Given a solution  $u$  and a time  $t$  so that  $u(t) \in \dot{B}_{\infty,\infty}^{-\epsilon}$ , let  $t' = t + T_*/2$  and  $t'' = t + T_*$  where  $T_*$  is as in (4) with  $a = u(t)$ . We now state and prove several (short) technical lemmas.

**Lemma 3.** *Fix  $\epsilon \in [0, 3/2)$  and  $T > 0$ . If  $u$  is a Leray-Hopf weak solution to 3D NSE on  $[0, T]$  and  $u(t) \in \dot{B}_{\infty,\infty}^{-\epsilon}$  for some  $t \in [0, T]$ , then for any  $M > 0$  we have*

$$\lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_\infty \leq M,$$

*provided*

$$j \leq \log_2 \left[ \left( c \frac{M}{\|u\|_{L^\infty(0,T;L^2)}} \right)^{2/(3-2\epsilon)} \right]$$

*for a suitable universal constant  $c$ .*

*Proof.* Assume  $u$  is a Leray-Hopf weak solution on  $[0, T]$  and  $t \in [0, T]$  such that  $\|u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}} < \infty$ . By Bernstein's inequalities we have

$$\|\dot{\Delta}_j u(t)\|_{\infty} \leq \lambda_j^{3/2} \|\dot{\Delta}_j u(t)\|_2.$$

Since  $u \in L^\infty(0, T; L^2) = L^\infty(0, T; \dot{B}_{2,2}^0)$ , for any  $j \in \mathbb{Z}$ ,

$$\lambda_j^{-\epsilon} \|\dot{\Delta}_j u\|_{\infty} \leq c \lambda_j^{3/2-\epsilon} \|u\|_{L^\infty(0,T;L^2)}.$$

Let

$$J(t) = \log_2 \left[ \left( \frac{M}{c \|u\|_{L^\infty(0,T;L^2)}} \right)^{2/(3-2\epsilon)} \right];$$

then

$$\sup_{j \leq J} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u\|_{\infty} \leq M.$$

□

**Lemma 4.** Fix  $\epsilon \in (0, 1)$  and  $T > 0$ , and assume  $u$  is a Leray-Hopf weak solution to 3D NSE on  $[0, T]$  belonging to  $C(0, T; \dot{B}_{\infty,\infty}^{-\epsilon})$ . Then, for any  $t_1 \in (0, T)$  and all  $t \in [t'_1, t''_1]$  we have

$$\sup_{\{j \in \mathbb{Z}: j \leq J_{low} \text{ or } j \geq J_{high}\}} \|\dot{\Delta}_j u(t)\|_{L^\infty} \leq \frac{1}{2} \|u(t_1)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}},$$

where  $J_{high}$  and  $J_{low}$  are defined by (1) and (2).

*Proof.* Using subcritical local well-posedness in  $\dot{B}_{\infty,\infty}^{-\epsilon}$  at  $t_1$  we have that there exists a mild/strong solution  $v$  defined on  $[t_1, t''_1]$ . By (6) we have

$$(t - t_1)^{1/2} \|v(t)\|_{\dot{B}_{\infty,\infty}^{1-\epsilon}} \leq c_0 \|v(t_1)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}$$

for all  $t \in (t_1, t''_1)$ . Since  $v(t_1) = u(t_1) \in L^2$  and since the strong solution  $v$  is smooth, integration by parts verifies that  $v$  is also a Leray-Hopf weak solution to 3D NSE. The weak-strong uniqueness result of [8] then guarantees that  $u = v$  on  $[t_1, t''_1]$ . Thus, for any  $t \in [t'_1, t''_1]$ ,

$$\lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_{\infty} \leq c \lambda_j^{-1} \|u(t_1)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}^{1/(1-\epsilon)+1}$$

for all  $j \in \mathbb{Z}$ . By (1) we conclude that

$$\sup_{j \geq J_{high}} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_{\infty} \leq \frac{1}{2} \|u(t_1)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}.$$

The low modes are eliminated using Lemma 3 with  $M = \|u(t_1)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}/2$ . □

**Definition 5.** We say that  $t$  is an escape time if there exists some  $M > 0$  such that  $t = \sup\{s \in (0, T) : \|u(s)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}} < M\}$ .

**Lemma 6.** Fix  $\epsilon \in (0, 1)$  and  $T > 1$ , and assume  $u$  is a Leray-Hopf weak solution to 3D NSE on  $[0, T]$  belonging to  $C(0, T; \dot{B}_{\infty, \infty}^{-\epsilon})$ . Let  $\mathcal{E}$  denote the collection of escape times in  $(0, T)$  and let  $I = \cup_{t \in \mathcal{E}} (t', t'')$ . Then

$$\int_0^T \|u(t)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}}^{2/(1-\epsilon)} dt = \infty, \quad (7)$$

if and only if

$$\int_I \|u(t)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}}^{2/(1-\epsilon)} dt = \infty. \quad (8)$$

*Proof.* It is obvious that (8) implies (7).

Assume (7). Let  $\{t_k\}_{k \in \mathbb{N}} \subset (0, T)$  be an increasing sequence of escape times which converge to  $T$  at  $k \rightarrow \infty$ . Clearly  $\|u(t_k)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}}$  blows up as  $k \rightarrow \infty$ . Since  $u \in C(0, T; \dot{B}_{\infty, \infty}^{-\epsilon})$ ,  $\|u(t_{k_1})\|_{\dot{B}_{\infty, \infty}^{-\epsilon}} < \|u(t_{k_2})\|_{\dot{B}_{\infty, \infty}^{-\epsilon}}$  for all  $k_1 < k_2$ .

We have two cases depending on the condition

$$\exists t_{k_0} \in \{t_k\} \text{ such that } \forall k \geq k_0 \text{ we have } t'_{k+1} \leq t''_k. \quad (9)$$

Case 1: If (9) is true, then  $[t'_0, T) = \cup_{k \geq k_0} [t'_k, t''_k)$ . In this case let  $I = [t'_0, T)$ . Clearly

$$\int_I \|u(t)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}}^{2/(1-\epsilon)} dt = \infty.$$

Case 2: If (9) is false then there exists an infinite sub-sequence of  $\{t_k\}$ , which we label  $\{s_k\}$ , such that  $s''_k < s'_{k+1}$  for all  $k \in \mathbb{N}$ . In this case let  $I = \cup_{k \in \mathbb{N}} [s'_k, s''_k)$ . Then,

$$\int_I \|u(t)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}}^{2/(1-\epsilon)} dt \geq \sum_{k \in \mathbb{N}} \frac{T^*(s_k)}{2} \|u(s_k)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}}^{2/(1-\epsilon)} = \sum_{k \in \mathbb{N}} \frac{c_0^{2/(1-\epsilon)}}{2} = \infty.$$

In either case, we have shown that (7) implies (8).  $\square$

### 3 Proofs of Theorem 1 and Theorem 2

*Proof of Theorem 1.* Fix  $\epsilon \in (0, 1)$  and  $T > 0$ , and assume  $u \in C(0, T; \dot{B}_{\infty, \infty}^{-\epsilon})$  is a Leray-Hopf weak solution to 3D NSE on  $[0, T]$ . Assume  $t_0, \dots, t_k$  are as in the statement of the lemma. It suffices to show

$$\|u(t_k)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}} \leq \|u(t_0)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}},$$

since then we re-solve at  $t_0$  and, by local-in-time well-posedness and the weak-strong uniqueness of [8], see that  $u$  is regular at time  $T$ .

If  $k = 0$ , then we are done. Otherwise note that  $t_1 \in (t'_0, t''_0)$ . Apply Lemma 4 at  $t_0$  to conclude that

$$\|u(t_1)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}} \leq \|u(t_0)\|_{\dot{B}_{\infty, \infty}^{-\epsilon}}.$$

If  $k = 1$ , then we are done. Otherwise, we repeat the argument and eventually obtain

$$\|u(t_k)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}} \leq \|u(t_0)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}},$$

which completes the proof.  $\square$

*Proof of Theorem 2.* Assume  $u$  is a Leray-Hopf weak solution on  $[0, T]$  which belongs to  $C(0, T; \dot{B}_{\infty,\infty}^{-\epsilon})$ .

By Lemma 3 with  $M = \|u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}/2$  it follows that

$$\sup_{j \leq J_{low}(t)} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_{\infty} < \frac{1}{2} \|u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}. \quad (10)$$

If  $u$  loses regularity at time  $T$ , local well-posedness in  $\dot{B}_{\infty,\infty}^{-\epsilon}$  implies that

$$\|u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}} \geq \left( \frac{c_*}{T-t} \right)^{(1-\epsilon)/2},$$

for a small universal constant  $c_*$ . Therefore,

$$\int_0^T \|u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}^{2/(1-\epsilon)} dt = \infty.$$

Let  $\mathcal{E}$  denote the collection of escape times in  $(0, T)$  and let  $I = \cup_{t \in \mathcal{E}} (t', t'')$ . By Lemma 6

$$\int_I \|u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}^{2/(1-\epsilon)} dt = \infty.$$

For each  $t \in I$  there exists an escape time  $t_0(t)$  so that  $t \in (t'_0, t''_0)$ . Thus,

$$\frac{1}{2} \left( \frac{c_0}{\|u(t_0)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}} \right)^{2/(1-\epsilon)} \leq t - t_0 \leq \left( \frac{c_0}{\|u(t_0)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}} \right)^{2/(1-\epsilon)}.$$

By re-solving at  $t_0$  using subcritical well-posedness, inequality (6), and weak-strong uniqueness (see [8]), we have

$$(t - t_0)^{1/2} \|u(t)\|_{\dot{B}_{\infty,\infty}^{1-\epsilon}} \leq c_0 \|u(t_0)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}.$$

Consequently,

$$\lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_{\infty} \leq 2c_0 \lambda_j^{-1} \|u(t_0)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}^{1+1/(1-\epsilon)} \leq 2c_0 \lambda_j^{-1} \|u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}^{1+1/(1-\epsilon)},$$

where we have used the fact that  $t_0$  is an escape time. Using (1) we obtain

$$\sup_{j \geq J_{high}(t)} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_{\infty} < \frac{\|u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}}{2}. \quad (11)$$

Combining (10) and (11) yields

$$\int_I \left( \sup_{J_{low}(t) \leq j \leq J_{high}(t)} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_{L^\infty} \right)^{2/(1-\epsilon)} dt = \infty,$$

which proves Theorem 2.  $\square$

**Remark 7.** If we only wanted to eliminate low frequencies in Theorem 2 then an alternative proof is available which we presently sketch. Decompose  $[0, T]$  into adjacent, disjoint intervals  $[t_k, t_{k+1})$  with  $t_{k+1} - t_k \sim 2^{-k}T$ . Then, a solution which is singular at  $T$  must satisfy

$$2^k \lesssim \|u(t \sim t_k)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}^{2/(1-\epsilon)}.$$

Using the Bernstein inequalities we have

$$\begin{aligned} \int_{t_k}^{t_{k+1}} \left( \sup_{j \leq J_0(t)} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_{L^\infty} \right)^{2/(1-\epsilon)} dt &\leq \int_{t_k}^{t_{k+1}} \left( \sup_{j \leq J_0(t)} \lambda_j^{3/2-\epsilon} \|\dot{\Delta}_j u(t)\|_2 \right)^{2/(1-\epsilon)} dt \\ &\lesssim \|u\|_{L^\infty L^2}^{2/(1-\epsilon)} \lambda_{J_0(t)}^{(3-2\epsilon)/(1-\epsilon)} (t_{k+1} - t_k) \\ &\lesssim \|u\|_{L^\infty L^2}^{2/(1-\epsilon)} 2^{J_0(t)(3-2\epsilon)/(1-\epsilon)} 2^{-k}. \end{aligned}$$

Define  $J_0$  so that  $J_0(t)(3-2\epsilon)/(1-\epsilon) = k/2$  for  $t \in [t_k, t_{k+1})$ . Then, terms on the right hand side are summable and we obtain

$$\int_0^T \left( \sup_{j \leq J_0} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_{L^\infty} \right)^{2/(1-\epsilon)} dt < \infty.$$

Since the integral over all modes must be infinite at a first singular time, we conclude

$$\int_0^T \left( \sup_{j \geq J_0(t)} \lambda_j^{-\epsilon} \|\dot{\Delta}_j u(t)\|_{L^\infty} \right)^{2/(1-\epsilon)} dt = \infty.$$

Further analyzing the definition of  $J_0$  and the lower-bound for the  $\dot{B}_{\infty,\infty}^{-\epsilon}$  norm we see that

$$J_0(t) \sim \log_2 \left( \|u(t)\|_{\dot{B}_{\infty,\infty}^{-\epsilon}}^{2/(3-2\epsilon)} \right),$$

which matches the rate found using the other approach.

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